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Stationary Axisymmetric Black Holes, $N = 2$ Superstring, and Self–Dual Gauge or Gravity Fields [1]

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Abstract

We present interesting relationship between what are called stationary axisymmetric black hole solutions for the vacuum Einstein equations in the ordinary four-dimensions and exact solutions for self-dual Yang-Mills fields in flat $2+2$ dimensions which are nothing but the consistent backgrounds for $N = 2$ open superstring. We show that any stationary axisymmetric black hole solution for the former automatically provides an exact solution for the latter. We also give a nice relation between the physical parameters of black holes and an invariant integral analogous to instanton charges for such a self-dual Yang-Mills solution. This result indicates that any general black hole solution in the usual four-dimensions can be the background of $N = 2$ superstring at the same time. We also give an interesting embedding of stationary axisymmetric solutions into the background of $N = 2$ closed superstring. Finally we show some indication that the Kerr solution with naked singularity (for $a > m$) is nothing else than the gravitational field generated by a closed string.

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1. Introduction

The physical as well as mathematical significance of $N = 2$ superstring theory [1] manifests itself in various contexts. First of all, the consistent background for this theory is supposed to be either self-dual Yang-Mills (SDYM) field for *open* $N = 2$ superstring or self-dual gravity for *closed* $N = 2$ superstring [2][3]. The physical importance of the SDYM field comes from the mathematical conjecture [4] that the SDYM theory may be the underlying theory of *all* lower-dimensional integrable theories, which are extremely important for many physical models. In a recent series of papers [5][6] we have also shown the generalization of this conjecture to the case of self-dual *supersymmetric* YM and self-dual supergravity (SDSG) theories.

It is then not surprising that some exact solutions for Einstein equation in general relativity, which are also known to be or conjectured to be integrable [7], may be embedded into the SDYM theory in $2 + 2$ dimensions. Independent of the progress in this direction, it has been also recently pointed out that the stationary axisymmetric solutions for vacuum Einstein equation is completely separated by re-formulating the Ernst equation [8] and its associated linear system in terms of a non-autonomous Schlesinger-type dynamical system [9] .

Independently of these developments, there has been also some progress related to what is called Kerr solution [10] with a ring singularity in axionic dilaton gravity may be regarded as a soliton-like solution for the heterotic string theory [11]. Interestingly it was found that some limit of this solution near the singular ring coincides with an exact solution for the heterotic string [11]. The importance of the Kerr solution strongly indicates the significant relevance of more general axisymmetric black hole solutions to string theories.

Based on these recent developments, we present in this paper explicit relationships between what are called axisymmetric black hole exact solutions in general relativity in $1 + 3$ dimensions and the SDYM for the gauge group $GL(2, \mathbb{R})$ in $2 + 2$ dimensions which are nothing but the consistent backgrounds for $N = 2$ open superstring [2][3][12]. We show that any stationary axisymmetric black hole solution in $1 + 3$ dimensions can be at the same time an exact solution for SDYM fields in $2+2$ dimensions. We also show a nice embedding of static axisymmetric stationary black hole solutions into the backgrounds of $N = 2$ *closed* superstring [12]. We finally show that the Kerr solution with the parameters $a > m$ has the energy-momentum tensor with δ-function singularity on a finite ring, indicating that this solution describes the gravitational field generated by a closed string.

2. Embedding of Ernst Equation into SDYM

We start with the review of what is called Ernst equation [8] for stationary axisymmetric exact solutions for vacuum Einstein equations. It is well-known [13] that any stationary axisymmetric space-time

$$ds^2 = f^{-1}\left[e^{2k}(dz^2 + d\rho^2) + \rho^2 d\phi^2\right] - f(dt + \omega d\phi)^2 \tag{2.1}$$

can be determined by a complex Ernst potential $\mathcal{E}(z,\rho)$ satisfying

$$\left[\frac{\rho(\mathcal{E}\mathcal{E}^*)_z}{(\mathcal{E}+\mathcal{E}^*)^2}\right]_z + \left[\frac{\rho(\mathcal{E}\mathcal{E}^*)_\rho}{(\mathcal{E}+\mathcal{E}^*)^2}\right]_\rho = 0 \ , \qquad \left[\frac{i\rho(\mathcal{E}-\mathcal{E}^*)_z}{(\mathcal{E}+\mathcal{E}^*)^2}\right]_z + \left[\frac{i\rho(\mathcal{E}-\mathcal{E}^*)_\rho}{(\mathcal{E}+\mathcal{E}^*)^2}\right]_\rho = 0 \ , \quad (2.2b)$$

$$\left[\frac{i\rho(\mathcal{E}^{*2}\mathcal{E}_z - \mathcal{E}^2\mathcal{E}^*_z)}{(\mathcal{E}+\mathcal{E}^*)^2}\right]_z + \left[\frac{i\rho(\mathcal{E}^{*2}\mathcal{E}_\rho - \mathcal{E}^2\mathcal{E}^*_\rho)}{(\mathcal{E}+\mathcal{E}^*)^2}\right]_\rho = 0 \ . \quad (2.2b)$$

In our notation the suffices $_z$ or $_\rho$ indicate the partial differentiations with respect to these variables, and the stars are for complex conjugates. Eq. (2.2b) is a necessary condition of the first two, and if the stationary axisymmetric solution is further specified to be *static*, the Ernst potential becomes real and (2.2b) is redundant.

In order to embed the Ernst equations into the SDYM fields in flat $2+2$ dimensions, we need to choose a convenient frame in the latter, specified by the coordinates $(x^\mu) = (z,\rho,\zeta,\tau)$ and the 4×4 flat metric [14]

$$(\eta_{\mu\nu}) = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \ , \tag{2.3}$$

where σ_3 is the third Pauli matrix. This new flat $2+2$ dimensional space-time is entirely different from the original $1+3$ dimensions (2.1). In the $2+2$ dimensions, the self-duality (SD) of the YM field: $F_{\mu\nu} = (1/2)\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$ is equivalent to the set of three equations[3]

$$F_{12}{}^I = 0 \ , \quad F_{34}{}^I = 0 \ , \quad F_{13}{}^I = F_{24}{}^I \ . \tag{2.4}$$

The indices $_{I,\ J,\ \cdots}$ are for the adjoint representation of a YM gauge group, which will be sometimes suppressed like (2.5).

Motivated by the 2×2 real matrix representation for the Ernst equation [13], we choose natural YM gauge group to be $GL(2,\mathbb{R})$, specifying the YM potential as

$$\begin{aligned} A_1 &= +Q\Sigma_0 + \tfrac{1}{\sqrt{2}}S(\Sigma_1+\Sigma_2) \ , \qquad A_2 = -P\Sigma_0 - \tfrac{1}{\sqrt{2}}R(\Sigma_1+\Sigma_2) \ , \\ A_3 &= +\widehat{S}\Sigma_0 - \tfrac{1}{\sqrt{2}}\widehat{Q}(\Sigma_1+\Sigma_2) \ , \qquad A_4 = -\widehat{R}\Sigma_0 + \tfrac{1}{\sqrt{2}}\widehat{P}(\Sigma_1+\Sigma_2) \ , \end{aligned} \tag{2.5}$$

[3] It is to be stressed that even though the first two equations imply that the components of the YM gauge field for these two-dimensional sub-manifolds are pure gauge, the gauge field may *not* be globally gauged away like monopoles with non-trivial topology. We will see explicit examples shortly.

where P, Q, R, S are real function only of (z, ρ), while $\widehat{P}, \widehat{Q}, \widehat{R}, \widehat{S}$ are real functions only of (ζ, τ), and Σ_I $(I = 0, \cdots, 3)$ are the generators for $GL(2, \mathbb{R})$:

$$\Sigma_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \qquad \Sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \qquad \Sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \qquad \Sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{2.6}$$

By this assignment, we are constructing the $2+2$ dimensions as a product of two manifolds with the coordinates (z, ρ) and (ζ, τ). We now see that the SD conditions (2.4) implies

$$P_z + Q_\rho = 0 \quad , \qquad R_z + S_\rho = 0 \quad , \tag{2.7a}$$

$$\widehat{P}_\zeta + \widehat{Q}_\tau = 0 \quad , \qquad \widehat{R}_\zeta + \widehat{S}_\tau = 0 \quad . \tag{2.7b}$$

The simplest assignment of these functions to embed the Ernst equations (2.2) is

$$P = \frac{\rho\,(\mathcal{E}\mathcal{E}^*)_z}{(\mathcal{E}+\mathcal{E}^*)^2} \quad , \qquad Q = \frac{\rho\,(\mathcal{E}\mathcal{E}^*)_\rho}{(\mathcal{E}+\mathcal{E}^*)^2} \quad , \qquad R = \frac{i\rho\,(\mathcal{E}-\mathcal{E}^*)_z}{(\mathcal{E}+\mathcal{E}^*)^2} \quad , \qquad S = \frac{i\rho\,(\mathcal{E}-\mathcal{E}^*)_\rho}{(\mathcal{E}+\mathcal{E}^*)^2} \quad . \tag{2.8}$$

Relevantly it is convenient to define

$$T = \frac{i\rho(\mathcal{E}^{*\,2}\mathcal{E}_z - \mathcal{E}^2\mathcal{E}^*_z)}{(\widetilde{\mathcal{E}} + \widetilde{\mathcal{E}}^*)^2} \quad , \qquad U = \frac{i\rho(\mathcal{E}^{*\,2}\mathcal{E}_\rho - \mathcal{E}^2\mathcal{E}^*_\rho)}{(\widetilde{\mathcal{E}} + \widetilde{\mathcal{E}}^*)^2} \quad . \tag{2.9}$$

We assign similarly $\widehat{P}, \widehat{Q}, \widehat{R}, \widehat{S}, \widehat{T}, \widehat{U}$ with (z, ρ) replaced by (ζ, τ) and the *hatted* Ernst potential as a function only of (ζ, τ), *e.g.*, $\widehat{P} \equiv \tau(\widehat{\mathcal{E}}\widehat{\mathcal{E}}^*)_\zeta/(\widehat{\mathcal{E}} + \widehat{\mathcal{E}}^*)^2$. We have thus a parallel structure between the two sets of coordinates (z, ρ) and (ζ, τ), so that the $2+2$ dimensions are now decomposed into two sets of Ernst equation systems each for two sub-dimensions of stationary axisymmetric solutions. There are lots of other equally simple assignments, but our choice here is such that an invariant integral (3.1) will have a certain manifest global symmetry, as we will see later.

We mention that any known instanton solutions for the Euclidean SDYM theory can be "Wick-rotated" to this $2+2$ dimensions. The BPST single $SU(2)$ instanton solution [15] in $4+0$ dimensions with the instanton number $k = 1$ can be re-casted into our $2+2$ dimensions for the non-compact gauge group $SL(2, \mathbb{R}) \subset GL(2, \mathbb{R})$:

$$\begin{aligned} A_1 &= 2\,[\,+\zeta\Sigma_3 + \tau(\Sigma_1 - \Sigma_2)\,]\,G \quad , \qquad A_2 = 2\,[\,+\tau\Sigma_3 - \zeta(\Sigma_1 + \Sigma_2)\,]\,G \quad , \\ A_3 &= 2\,[\,-z\Sigma_3 + \rho(\Sigma_1 + \Sigma_2)\,]\,G \quad , \qquad A_4 = 2\,[\,-\rho\Sigma_3 + z(\Sigma_1 - \Sigma_2)\,]\,G \quad , \end{aligned} \tag{2.10}$$

where $G \equiv 1/(x_\mu x^\mu + b^2)$ with an arbitrary constant b, and $x_\mu x^\mu = 2(z\zeta - \rho\tau)$ with the metric (2.3).

We can superimpose this instanton solution onto the above SDYM fields constructed from black hole solutions, as long as the commutators between them vanish. Moreover, if we have a new set of solutions for the SDYM in $2+2$ dimensions, we may "Wick rotate" it to the Euclidean $4+0$ dimensions.

3. Black Hole Parameters and Invariant Integral

Once we have succeeded in embedding the stationary axisymmetric solutions into SDYM in $2+2$ dimensions, we can consider "instanton charge" for a SDYM field. Since our space-time and the YM gauge group are non-compact, there is no strict concept such as the topological instanton charge for the SDYM fields. Nevertheless we still can formally define an analogous invariant integral:

$$C \equiv \frac{1}{64\pi^2} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^I F_{\rho\sigma}{}^I \quad , \tag{3.1}$$

as a direct analog of the usual instanton charge. For examples below we restrict the YM group to be $GL(2, \mathbb{R})$.

The integrand in (3.1) is a total-divergence, and the only contributions come from the surface terms. We can evaluate this charge for any known exact solutions such as the Kerr solution [10] or Tomimatsu-Sato (TS) solutions [16]. To this end, it is convenient to use the prolate spherical coordinates (x, y):

$$z \equiv \kappa x y \quad , \qquad \rho \equiv \kappa \sqrt{(x^2-1)(1-y^2)} \quad . \tag{3.2}$$

The constant κ has the physical dimension of length. We can now rewrite (3.1) in terms of the surface integrals:

$$C[\mathcal{E}, \widehat{\mathcal{E}}] = \frac{1}{8\pi^2} \Big(X[P,Q]\, X[\widehat{R}, \widehat{S}] - X[R,S]\, X[\widehat{P}, \widehat{Q}] \Big) \quad , \tag{3.3}$$

where[4]

$$\begin{aligned} X[P,Q] &\equiv \int dx^1 \int dx^2 (P_z + Q_\rho) \\ = &- \kappa \int_{-1}^{1} dy \left[\lim_{x\to\infty} (xP) \frac{y}{\sqrt{1-y^2}} - \lim_{x\to 1} (\sqrt{x-1}\, P) \frac{\sqrt{2}y}{\sqrt{1-y^2}} + \lim_{x\to\infty} (xQ) - \lim_{x\to 1} Q \right] \\ &- \kappa \int_{1}^{\infty} dx \left[\left\{ \lim_{y\to 1} (\sqrt{1-y}\, P) - \lim_{y\to -1} (\sqrt{1+y}\, P) \right\} \frac{\sqrt{2}x}{\sqrt{x^2-1}} - \lim_{y\to 1} Q - \lim_{y\to -1} Q \right] \quad . \end{aligned} \tag{3.4}$$

[4]We consider only the connected patch of each coordinate system for the integral.

Another quantity $X[R,S]$ is simply obtained by replacing (P,Q) in (3.4) by (R,S). It is clear now that our invariant integral is a product of surface integrals like monopole charges living on the two-dimensional sub-manifolds.

We can develop a more convenient formula for $X[P,Q]$, when some asymptotic form of the Ernst potential is known. Suppose at $x \approx \infty$ we have

$$\mathcal{E} \approx 1 + \frac{F(y)}{x} + \mathcal{O}(x^{-2}) \quad , \tag{3.5}$$

with a complex function $F(y)$ of y. The first term of (3.5) is general enough for the boundary condition $\mathcal{R}e\,\mathcal{E} \rightarrow 1$ for asymptotically flat solutions, because the Ernst equation (2.2) is invariant under the shift of $\mathcal{E}$ by arbitrary purely imaginary constant [13]. If we define $G(y) \equiv \mathcal{R}e\,F(y)$, $H(y) \equiv \mathcal{I}m\,F(y)$, these functions are actually determined by the Ernst eqs. (2.7) themselves at this order:

$$G(y) = a\ln\left(\frac{1-y}{1+y}\right) + G_0 \quad , \qquad H(y) = c\ln\left(\frac{1-y}{1+y}\right) + H_0 \quad , \tag{3.6}$$

where a, c, G_0, H_0 are constants. The asymptotic form of the g_{00}-component of the original line element (2.1) at $z \approx 0$ and $\rho \approx \infty$ namely $y \approx 0$, $x \approx \infty$ must be

$$g_{00} = -f = -\mathcal{R}e\,\mathcal{E} \approx -1 - \frac{G_0}{x} + \mathcal{O}(x^{-2}) \approx -1 + \frac{2m}{\rho} + \mathcal{O}(\rho^{-2}) \quad . \tag{3.7}$$

to accord with the Newtonian approximation. This requires the residue G_0 to be

$$G_0 = -\frac{2m}{\kappa} \quad . \tag{3.8}$$

The asymptotic forms of $P, \cdots, S$ can be also fixed. Using (3.5), we can easily see that

$$\begin{aligned}
P &\approx -\frac{1}{2x}\sqrt{1-y^2}\left[yG - (1-y^2)G_y\right] + \mathcal{O}(x^{-2}) \quad , \\
Q &\approx -\frac{1}{2x}(1-y^2)\left(G + yG_y\right) + \mathcal{O}(x^{-2}) \quad , && (3.9a) \\
R &\approx +\frac{1}{2x}\sqrt{1-y^2}\left[\,yH - (1-y^2)H_y\right] + \mathcal{O}(x^{-2}) \quad , \\
S &\approx +\frac{1}{2x}(1-y^2)(H + yH_y) + \mathcal{O}(x^{-2}) \quad . && (3.9b)
\end{aligned}$$

Without much loss of generality[5] we can also assume that

$$\lim_{x\rightarrow 1}(\sqrt{x-1}\,P) = \lim_{y\rightarrow\pm 1}(\sqrt{1\mp y}\,P) = \lim_{x\rightarrow 1} Q = \lim_{y\rightarrow\pm 1} Q = 0 \quad , \tag{3.10}$$

[5]This property seems common to asymptotically free black hole solutions.

and *idem.* for R and S. We now see that only two terms $\lim_{x\to\infty}(xP)$ and $\lim_{x\to\infty}(xQ)$ in (3.4) remain under (3.6) and (3.9), therefore X's is completely determined by the residues

$$X[P,Q] = \frac{\kappa}{2}\int_{-1}^{1} dy \; G(y) = \kappa G_0 = -2m \quad , \qquad X[R,S] = \frac{\kappa}{2}\int_{-1}^{1} dy \; H(y) = -\kappa H_0 \quad , \tag{3.11}$$

and we can easily estimates the invariant integral for arbitrary exact solutions:

$$C[\mathcal{E},\widehat{\mathcal{E}}] = \frac{\kappa^2}{8\pi^2}\left(G_0\widehat{H}_0 - H_0\widehat{G}_0\right) \quad . \tag{3.12}$$

As is easily seen, there are many options for the choices of the Ernst potentials $\mathcal{E}$ and $\widehat{\mathcal{E}}$. For example, we can choose $\mathcal{E}$ to be the Kerr solution, while $\widehat{\mathcal{E}}$ to be a TS solution. We show below relevant residues needed for representative cases of the Kerr solution with $\delta = 1$ [10] (Kerr$^{\delta=1}$ for short), Weyl solution (Weyl$^{\delta}$), and the $\delta = 2$ case of the TS solution [16] (TS$^{\delta=2}$):

(i) Kerr Solution of $\delta = 1$:

$$\begin{aligned} &\mathcal{E} = \frac{\alpha-\beta}{\alpha+\beta} \quad , \qquad \alpha \equiv px - iqy \quad , \qquad \beta \equiv 1 \qquad (p^2+q^2=1) \quad , \\ &\kappa G_0 = -\frac{2\kappa}{p} = -2m \quad , \qquad \kappa H_0 = 0 \quad . \end{aligned} \tag{3.13}$$

(ii) Weyl solution with general δ:

$$\mathcal{E} = \frac{(x-1)^\delta}{(x+1)^\delta} \quad , \qquad \kappa G_0 = -2\delta\kappa = -2m \quad , \qquad \kappa H_0 = 0 \quad . \tag{3.14}$$

(iii) TS solution for $\delta = 2$:

$$\begin{aligned} &\mathcal{E} = \frac{\alpha-\beta}{\alpha+\beta} \;, \quad \alpha \equiv p^2x^4 + q^2y^4 - 1 - 2ipqxy(x^2-y^2) \;, \quad \beta \equiv 2px(x^2-1) - 2iqy(1-y^2) \;, \\ &\kappa G_0 = -\frac{4\kappa}{p} = -2m \quad , \qquad \kappa H_0 = 0 \qquad (p^2+q^2=1) \quad . \end{aligned} \tag{3.15}$$

If we choose any of these solutions for $\mathcal{E}$ and $\widehat{\mathcal{E}}$, the invariant integral (3.12) vanishes due to $H_0 = 0$. We will give an example of non-vanishing cases shortly.

The solutions for the Ernst equation (2.2) have a global $SL(2,\mathbb{R})$ symmetry [13]:

$$\widetilde{\mathcal{E}} = \frac{a\mathcal{E} - ib}{ic\mathcal{E} + d} \qquad (ad - bc = 1) \quad , \tag{3.16}$$

where the constants a, b, c, d are real, and all the *tilded* quantities in this section are after a global $SL(2,\mathbb{R})$ transformation. The asymptotic condition $\mathcal{R}e\,\widetilde{\mathcal{E}} \to 1 \; (x\to\infty)$ yields

$$c^2 + d^2 = 1 \quad , \tag{3.17}$$

due to $\mathcal{E} \to 1$ by (3.5). The transformation (3.16) is rewritten in terms of $P, \cdots, U$:

$$\begin{aligned} \widetilde{P} &= (ad+bc)P + bdR - acT \quad , \qquad \widetilde{Q} = (ad+bc)Q + bdS - acU \quad , \\ \widetilde{R} &= 2cdP + d^2R - c^2T \quad , \qquad \widetilde{S} = 2cdQ + d^2S - c^2U \quad , \\ \widetilde{T} &= -2abP - b^2R + a^2T \quad , \qquad \widetilde{U} = -2abQ - b^2S + a^2U \quad . \end{aligned} \tag{3.18}$$

Under this global symmetry, the different components of the YM field strength (2.5) are transformed into each other. Applying this to (3.9), we see that the residues transform as

$$\widetilde{G}_0 = (ad+bc)G_0 + (ac-bd)H_0 \quad , \qquad \widetilde{H}_0 = -2cdG_0 + (d^2-c^2)H_0 \quad . \tag{3.19}$$

Under (3.17), this linear transformation preserves its determinant to be unity. This signals the existence of a bilinear form of $G_0\widehat{H}_0 - H_0\widehat{G}_0$ invariant under such transformations in the solution space, and this is exactly the combination we had in our invariant integral (3.3). In other words, our embedding rule (2.5) was chosen in such a way that the integral (3.1) or (3.12) is invariant under the global $SL(2,\mathbb{R})$ symmetry for asymptotically flat solutions.

The invariance of the integral (3.9) under the global $SL(2,\mathbb{R})$ symmetry can be utilized to test whether two exact solutions are linked to each other under such global transformations. For example, we can see if the $\mathrm{TS}^{\delta=2}$ solution is connected to the $\mathrm{Weyl}^{\delta=2}$ solution by comparing invariant integrals: $C_1[\mathcal{E}_1, \widehat{\mathcal{E}}_1]$ and $C_2[\mathcal{E}_2, \widehat{\mathcal{E}}_2]$ where $\mathcal{E}_1 = \mathrm{Weyl}^{\delta=1}$, $\widehat{\mathcal{E}}_1 = \widetilde{\mathrm{Weyl}}^{\delta=2}$, $\mathcal{E}_2 = \mathrm{Weyl}^{\delta=1}$, $\widehat{\mathcal{E}}_2 = \mathrm{TS}^{\delta=2}$, where *tilde* implies the exact solutions after applying a transformation (3.16). This is because if $\mathrm{TS}^{\delta=2} = \widetilde{\mathrm{Weyl}}^{\delta=2}$ under such a transformation, we will get $C_1 = C_2$. By studying the values of parameters a, b, c, d that satisfy $C_1 = C_2$, we can see their connectedness for these special values. In fact, we easy see that they are not really connected, because $C_1 = 0$, $C_2 = (2m^2cd)/\pi^2$, and the inspection of the possible cases $c = 0$ and $d = 0$ reveals no connectedness. Even though this example is rather trivial, we can use this invariant integral to test a "newly" found exact solution, to see its connectedness with any known exact solution, whenever the direct comparison is difficult.

4. Stationary Axisymmetric Black Holes for Closed $N = 2$ Superstring

We have so far dealt with the SDYM fields for *open* $N = 2$ superstring. We may wonder about the SDSG for *closed* $N = 2$ superstring. In fact, we have already presented such embedding of the dilaton black hole solution [17] into the backgrounds [12] of *closed* $N = 2$ superstring [18]. Below we will show how the embedding of stationary axisymmetric vacuum solutions can be also embedded into such backgrounds.

The consistent background fields for closed $N = 2$ superstring form a multiplet of $N = 8$ SDSG in $2+2$ dimensions [12]. Among bosonic background fields, we require that

the fields $\phi_{ABCD},\ A_\mu{}^{AB},\ B_\mu{}^{AB}$ [12][6] are all zero for simplicity in our prescription. Eventually our non-vanishing fields are $(e_\mu{}^m, \omega_\mu{}^{rs})$,[7] where $\omega_\mu{}^{rs}$ is self-dual for the indices ^{rs}, and the relevant field equations are

$$2\epsilon^{\mu\nu\rho\sigma} D_\nu \Lambda_{\rho\sigma m} - \epsilon^{\mu\nu\rho\sigma} \Lambda_{\nu m}{}^n T_{\rho\sigma n} = 0 \quad , \tag{4.1a}$$

$$\epsilon^{\mu\nu\rho\sigma} \left[e_\nu{}^{[m|} T_{\rho\sigma}{}^{|n]} - \frac{1}{2} \epsilon^{mn}{}_{rs} e_\nu{}^r T_{\rho\sigma}{}^s \right] = 0 \quad . \tag{4.1b}$$

The torsion tensor is $T_{\mu\nu}{}^m \equiv 2(\partial_{[\mu} e_{\nu]}{}^m + \omega_{[\mu}{}^{mn} e_{\nu]n})$.

As the simplest solutions for (4.1) we have $\omega_\mu{}^{rs} = 0$ implying the lack of local Lorentz symmetry in the system, while the vierbein is specified as

$$(e_\mu{}^m) = \begin{pmatrix} 1-Q & S & 0 & 0 \\ P & 1-R & 0 & 0 \\ 0 & 0 & 1-\widehat{S} & \widehat{Q} \\ 0 & 0 & \widehat{R} & 1-\widehat{P} \end{pmatrix} \quad , \tag{4.2}$$

where we use (2.8), the flat metric η_{mn} of (2.3), and the previous rule for *hatted* fields. It is now straightforward to show that (4.2) yields $T_{\mu\nu}{}^r = 0$ similar to the previous case of SDYM in the flat space-time, and thereby all the field equations in (4.1) are satisfied.

This torsion is like an analog of the SDYM field, and we expect an invariant integral

$$C' \equiv c \int d^4x \ \epsilon^{\mu\nu\rho\sigma} T_{\mu\nu}{}^m T_{\rho\sigma m} = 8c \Big(X[P,Q] X[\widehat{R},\widehat{S}] - X[R,S] X[\widehat{P},\widehat{Q}] \Big) \quad , \tag{4.3}$$

which exactly coincides with (3.3) up to a constant: $C' =$ const. $\times\, C$. Even though the vierbein with vanishing anholonomy coefficients seems trivial, non-vanishing integral (4.3) indicates the non-trivial feature of this gravitational system with the connection fields that can not be globally gauged away. Like the previous SDYM, the embedding (4.2) is fixed in such a way that C' is invariant under the global $SL(2,\mathbb{R})$ group for asymptotically flat solutions, which can be used as an index to see the link between solutions.

The above example seems the simplest, but there may be other non-trivial embedding scenarios we can develop with more non-trivial vierbeins. In any case, it is now obvious that any stationary axisymmetric black hole solution in $1+3$ dimensions can be also a consistent background in $2+2$ dimensions for the closed as well as open $N=2$ superstring at the same time!

[6] We use the same notation as in ref. [18]. Consistency of such truncation can be confirmed by studying the original set of all the field equations [12], which we skip in this paper.

[7] The indices $\mu, \nu, \cdots$ and $m, n, \cdots$ are respectively the curved and local Lorentz coordinates in $2+2$ dimensions with the coordinates $(x^\mu) = (z, \rho, \zeta, \tau)$.

5. Kerr Solution as Gravity around Closed String

Our results so far can be also reinterpreted based on the philosophy that the $N = 2$ superstring is expected to be "Master Theory" [3][5] of all the other superstring theories. We have seen black hole solutions for general relativity (the backgrounds of $N = 1$ superstring in $1 + 3$ dimensions) are embedded into the background field equations of $N = 2$ superstring in $2 + 2$ dimensions. There is other supporting evidence for the $N = 2$ superstring to be the Master Theory of $N \leq 1$ (super)string theories, *e.g.,* we have shown [6] that $N = 1$ superconformal Wess-Zumino-Novikov-Witten sigma-models themselves are embedded into the SDYM fields as the consistent backgrounds of $N = 2$ superstring, indicating that $N = 1$ superstrings exist as the direct target space of $N = 2$ superstrings. If this is indeed the case, we can suspect that the black hole solutions in $1+3$ dimensions are nothing but the gravitational fields generated by $N \leq 1$ (super)strings themselves.[8]

Motivated by this observation, we try below to show that the the matter energy density for the Kerr solution [10] with $a > m$ has a manifest naked δ-function singularity at a finite radius on the equatorial plane. We start with the Kerr solution with the oblate spherical coordinates (u, y):

$$ds^2 = m^2 \left[-e^{2\nu} dt^2 + e^{2\psi} (d\varphi - \Omega dt)^2 + e^{2\mu_2} du^2 + e^{2\mu_3} dy^2 \right] \quad , \tag{5.1}$$

$$\Omega = \frac{2q^2(\widehat{p}u + 1)}{mD} \quad , \quad e^{2\nu} = \frac{\widehat{p}^2(u^2+1)B}{D} \quad , \quad e^{2\psi} = \frac{(1-y^2)D}{B} \quad ,$$

$$e^{2\mu_2} = \frac{B}{u^2+1} \quad , \quad e^{2\mu_3} = \frac{B}{1-y^2} \quad ,$$

$$B \equiv (\widehat{p}u+1)^2 + q^2y^2 \quad , \quad D \equiv \left[(\widehat{p}u+1)^2 + q^2 \right]^2 - \widehat{p}^2q^2(u^2+1)(1-y^2) \quad , \tag{5.2}$$

where ν, μ_2, μ_3, ψ are functions only of (u, y), and $q^2 = 1 + \widehat{p}^2$. We use the Kerr solution for the case $a > m$ analytically continued from the case $a \leq m$ with the prolate coordinates with $p^2 + q^2 = 1$. Now the singularity is *naked* in this Kerr solution and hence it really "exists" on the ring $\widehat{p}u + 1 = 0$, $y = 0$, where the Riemann tensor diverges. Following ref. [20], the 00-component of the matter energy-momentum tensor is computed *via* the Einstein tensor in the "locally non-rotating reference frame" as

$$\begin{aligned} 8\pi T_{(0)(0)} &= R_{(0)(0)} + \tfrac{1}{2}R = m^{-2}e^{-\psi-\mu_2-\mu_3} \left\{ \left[e^{\mu_3-\mu_2}(e^\psi)_u \right]_u - \left[e^{\mu_2-\mu_3}(e^\psi)_y \right]_y \right\} \\ &+ m^{-2}e^{-\mu_2-\mu_3} \left\{ \left[e^{-\mu_2}(e^{\mu_3})_u \right]_u - \left[e^{-\mu_3}(e^{\mu_2})_y \right]_y \right\} + \tfrac{1}{4}m^{-2}e^{2\psi-2\nu} \left[\Omega_u^2 e^{-2\mu_2} - \Omega_y^2 e^{-2\mu_3} \right] \quad . \end{aligned} \tag{5.3}$$

where the index ${}_{(0)}$ denotes the locally non-rotating 0-th coordinates, and subscripts ${}_{u,\ y}$ are for partial derivatives. Since the Kerr metric is a vacuum solution, (5.3) vanishes everywhere *except* the above naked ring singularity.

[8]This conjecture appears in many different contexts [11][19], but our motivation is stronger by our philosophy based on $N = 2$ superstring supported also by mathematics.

Our purpose is to show that this energy-momentum tensor component has a δ-function at the naked singularity, like $\delta(\widehat{p}u+1)\delta(y)$, because the energy-momentum tensor vanishes everywhere else as a vacuum solution, but at the same time its u and y-integral should give a non-vanishing mass. Showing this is generally difficult, because unless we have an appropriate "regulator" function, we will simply get a vanishing result. There have been also some trials such as disk like mass distributions [21] to explain the singularity, but none of them seems to give the *manifest* δ-function singularity. Our approach is also different from the interpretation by complex hyperbolic string in ref. [22].

A similar regulator is found in particle physics for a Coulomb potential $\phi(r)$ for the Laplace equation, which can be regularized by the Yukawa potential in three dimensions:

$$\Delta\phi(r) = \lim_{\beta\to 0}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\frac{e^{-\beta r}}{4\pi r} = \lim_{\beta\to 0}\left(\beta^2\,\frac{e^{-\beta r}}{4\pi r}\right) \quad , \tag{5.4}$$

yielding

$$\int_0^\infty dr\ 4\pi r^2\ \Delta\phi(r) = \lim_{\beta\to 0}\beta^2\int_0^\infty dr\left(re^{-\beta r}\right) = +1 \quad , \tag{5.5}$$

implying that the r.h.s. of (5.4) is actually a δ-function: $\delta(r)/(4\pi r^2)$ instead of 0. If we take the $\beta\to 0$ limit *before* the r-integral, we get simply zero with no singularity. This fundamental feature can be understood easily, but it is generally difficult to find a manifest regulator to show the singularity, especially when the system is non-linear.[9] Here we show an explicit working-example of such a regulator.

Our regulator acts only on the g_{uu} and g_{yy}-components of the metric:

$$e^{\widetilde{\mu}_2} = S(r)e^{\mu_2} \quad , \qquad e^{\widetilde{\mu}_3} = S(r)e^{\mu_3} \quad , \qquad S(r)\equiv 1+b\,\alpha e^{-\alpha r/m} \quad . \tag{5.6}$$

where the *tilded* quantities are regularized ones, and $S(r)$ is our regulator, which goes to unity as the parameter $\alpha\to 0$. The real constant b will be fixed shortly, and r is the Boyer-Lindquist radial coordinate: $r\equiv m(px+1)$ or $r\equiv m(\widehat{p}u+1)$. This $S(r)$ is an analog of $\exp(-\beta r)$ in (5.5). Due to the cancellation in the exponents and y-independence of $S(r)$, it is only the third term in (5.3) that will contribute.

We first fix the constant b in (5.6), applying it to the Schwarzschild metric. The spacial volume integral of (5.3) yields the non-zero result:

$$\lim_{\alpha\to 0}\int_0^\infty d\xi\int_{-1}^1 dy\int_0^{2\pi} d\varphi\sqrt{-\widetilde{g}}\,m\ \widetilde{T}^{00}\Big|_{\text{Schwarzschild}} = \frac{1}{2}b\,m \quad . \tag{5.7}$$

where $\xi\equiv r/m = x+1$. This implies the existence of the δ-function singularity

$$m^2\,T^{00}\Big|_{\text{Schwarzschild}} = \frac{bm}{8\pi r^2}\delta(r) \quad . \tag{5.8}$$

[9] We could easily rely on Fourier transforms or test functions, if the system were linear.

Due to the original dimensionless time coordinate, $m^2 T^{00}$ corresponds to the energy density. To accord with the standard point mass distribution [23], we fix

$$b = 2 \quad . \tag{5.9}$$

We mention the following important points to simplify the computation of extracting only $\mathcal{O}(\alpha^0)$ part of the total expression in (5.7):

(i) We perform a double-expansion for the integrand: $\sum_{k,n} A_{k,n}(y)\alpha^k \, (\xi+1)^n \exp(-l\alpha\xi)$ where the integer $l \; (l \geq 1)$ is fixed, and the $\exp(-l\alpha\xi)$ is *not* expanded.

(ii) The relevant terms can be estimated by the simple integrals of the type $\alpha^k \int_0^\infty d\xi \; (\xi+1)^n \exp(-l\alpha\xi)$ with $k \geq 1, \; n \leq 2$.

Some remarks are in order: For (i) we can expand around any negative value of ξ instead of -1, but the result does not depend on such a value: It is just to avoid the singularity at $\xi = 0$. For (ii) all such integrals for $n \leq -2$ with $\alpha^k \; (k \geq 1)$ in front are easily shown to vanish, while the subtle case $k \geq 1, \; n = -1$ can be also shown to be of the order $\alpha^k \ln[(\alpha+1)/\alpha]$ which again vanishes as $\alpha \rightarrow 0$. The only non-zero contributions in the above computation are the cases $(k, n) = (1, 0)$, $(2, 1)$ and $(3, 2)$, which are all finite.

Following these points, we apply our regulator (5.6) now to the Kerr solution (5.2):[10]

$$\lim_{\alpha \rightarrow 0} \frac{1}{\widehat{p}} \int_0^\infty d\xi \int_{-1}^{1} dy \int_0^{2\pi} d\varphi \; \sqrt{-g} \, m \; \widetilde{T}^{00} \Big|_{\text{Kerr}} = m \quad , \tag{5.10}$$

where now $\xi \equiv r/m = \widehat{p}u + 1$. This accords with the asymptotic total mass of the system [23]. Since the whole integrand vanishes *except* the singularity as the vacuum Kerr solution as $\alpha \rightarrow 0$, this implies the existence of a δ-function singularity:

$$m^2 \, T^{00} \Big|_{\text{Kerr}} = \frac{\widehat{p}}{2\pi\sqrt{-g}} \delta(\xi)\delta(y) = \frac{m^3}{2\pi a^2 (\rho - a)^2} \delta(\rho - a)\delta(\cos\theta) \quad , \tag{5.11}$$

in terms of the coordinates $z \equiv m\widehat{p}uy, \; \rho \equiv m\widehat{p}\sqrt{(u^2+1)(1-y^2)}$. This means the existence of such ring-like mass distribution around the circle at the finite radius of $\rho = a$ on the equatorial plane $\theta = \pi/2$, which is nothing but a "closed string"! It is interesting that we have obtained the normalized total mass m also for the Kerr solution, providing another support for the validity of our regulator (5.6). It seems that our regulator can be applied also to other axisymmetric solutions.

[10] The lower limit of the ξ-integral can be any finite non-positive number, as long as the integral includes the singularity at $\xi = 0$.

6. Concluding Remarks

In this paper we have presented an interesting relationship between what are called stationary axisymmetric black hole solutions and SDYM fields in $2+2$ dimensions, which are nothing but the consistent backgrounds for $N=2$ *open* superstring. We have confirmed that any stationary axisymmetric solution of the vacuum Einstein equation can be the backgrounds of $N=2$ open superstring at the same time! We have found interesting relations between the parameters for the axisymmetric black holes and the invariant integrals analogous to topological instanton charges. In particular, the invariant integral of any axisymmetric black hole solution is determined by the residues in the asymptotic expansion of the Ernst potential, like asymptotic masses of the black holes. We have also performed a nice embedding of these black hole solutions into the background vierbein of the $N=2$ *closed* superstring, and developed a similar invariant integral. We finally showed the indication that the Kerr solution for $a>m$ is nothing but the gravitational field around a closed string itself.

To our knowledge, our work is the first attempt to relate the gravitational exact solutions to the backgrounds for the $N=2$ superstring *via* SDYM and SDSG fields in an explicit way. As a by-product, our method also gives the series of new exact solutions for the SDYM field by superpositions of already-known instanton solutions such as the BPST solution onto SDYM fields constructed from a black hole solution.

Our invariant integrals both for the SDYM and SDSG fields have the manifest global $SL(2,\mathbb{R})$ symmetry acting on asymptotically flat solutions in addition to the local $GL(2,\mathbb{R})$ gauge group. Therefore we can also use this integral as a convenient index indicating the "connectedness" of two given solutions under such global transformations.

The clear relationship between the stationary axisymmetric black holes and $N=2$ superstring is also interesting, considering the recent development about the Kerr solution in the axion dilaton gravity coinciding with the background solutions for heterotic string [11]. The recent results relating conformal field theory and black hole solutions [9], or the interpretation of the ring singularity in the Kerr solution interpreted as a complex hyperbolic string [22], also support this philosophy. In contrast to the usual bosonic or $N=1$ superstring theory accommodating black hole solutions in $1+3$ dimensions [19], our result shows the direct link between $N=2$ superstring in $2+2$ dimensions and black holes in $1+3$ dimensions. We have thus provided another strong motivation of investigating black hole solutions in general relativity based on superstring physics.

Our explicit results also strongly support the philosophy that the $N=2$ superstring theory is really the Master Theory of all (supersymmetric) integrable systems in lower-dimensions [3][5]. It is reasonable that the background field equations of such Master Theory are of the first order, in particular the SD equations are the most natural first-order equa-

tions for bosonic fields. This is because lower-order differential equations are more universal as embedding equations in such a Master Theory. In our examples, we have seen that the black hole solutions for the second-order Einstein field equation in the $1+3$ dimensions are embedded into the first-order SDYM or SDSG equations for the backgrounds of $N=2$ superstrings in $2+2$ dimensions. Even though at first glance the embedding of exact solutions in $1+3$ dimensions into $2+2$ dimensions is rather bizarre due to the fundamental difference in signature, once we have understood the significance of the integrable systems governing the basic differential equations *via* self-dual theories, it becomes easy to comprehend the naturalness of such embeddings.

Motivated by this philosophy, we conjecture that any black hole solution in $1+3$ dimensions for general relativity is nothing but the gravitational field generated by $N \le 1$ (super)strings. This is because general relativity is the consistent background for $N \le 1$ superstring, which in turn is to be the target space for the $N=2$ superstring [6]. In fact, we have shown that the 00-component of the energy-momentum tensor for the Kerr solution with $a > m$ actually has the δ-funciton singularity on the ring at $\rho = a,\ \theta = \pi/2$, using a particular regulator for some metric components, which seems more applicable to other solutions. Our study provides strong evidence that string physics is playing important roles also for black hole physics.

Our results provide a completely new motivation for the observation of black holes in real space. So far string theory had been regarded as physics at "unrealistically" high energy around the Planck mass that can not to be easily realized by the present "low-energy" technology. However, thanks to the encouraging results connecting strings with black holes, string theory has now gained more realistic aspects that can be probed by the exploration of black holes in the real universe.

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